

一类随机 SICK 丙肝模型的 dynamics 分析

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摘要: 为了研究现实中不确定因素对丙肝的影响, 考虑了一类具有急性与慢性感染的随机 SICK 丙肝模型。首先, 依据停时理论证明了全局正解的存在唯一性; 其次, 通过构造 Lyapunov 函数并结合伊藤公式, 讨论了随机模型的解在确定模型的无病平衡点和正平衡点附近的振荡行为, 并给出了随机模型解的平均持续和疾病灭绝的充分条件。最后, 数值模拟考虑了噪声对模型的影响, 结果表明: 随机模型在确定模型的平衡点附近扰动, 扰动强度与噪声强度成正比, 并且足够大的噪声使疾病灭绝, 进一步证实了理论结果。

关键词: 随机模型; 平衡点; 振荡行为; 伊藤公式; 疾病持续与灭绝

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Dynamic Analysis of an SICK Hepatitis C Infection Model with Stochastic Effects

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Abstract: In order to study the influence of uncertain factors on hepatitis C in reality, a randomized SICK hepatitis C model with acute and chronic infection was considered. Firstly, the existence and uniqueness of global positive solutions are proved according to the stopping time theory; secondly, by constructing Lyapunov function and combining Ito formula, the oscillation behavior of the solution of the stochastic model near the disease-free equilibrium point and the positive equilibrium point of the model is discussed, and then the sufficient conditions for the average persistence and disease extinction of the solution of the stochastic model are given. Finally, the numerical simulation considers the influence of noise on the model. The results show that the stochastic model is disturbed near the equilibrium point of the determined model, and the disturbance intensity is positively correlated with the noise intensity. Moreover, the noise is large enough to make the disease extinct, which further confirms the theoretical results.

Keywords: stochastic model; equilibrium point; oscillatory behavior; Ito formula; disease persistence and extinction

0 引言

丙肝是由丙型肝炎病毒 (hepatitis C virus, HCV) 引起的肝脏炎症, 是一种全球性流行病。根

据世界卫生组织 2018 年统计, 全球 HCV 的感染率约为 3%, 约 1.8 亿人感染了丙肝。丙肝患者分为急性与慢性阶段, 大多数急性患者症状不明显, 少数患者会自发清除病毒产生免疫力, 未经抗病毒治疗的急性丙肝患者大约 80% 会转为慢性感染者, 发展

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为慢性感染的患者可导致肝脏慢性炎症坏死和纤维化,一部分可能发展为肝硬化甚至肝细胞癌^[1-3]。

近年来,国内外许多学者对丙肝的传播机理进行了深入的研究^[4-7]。文[8]中 Martcheva 等引入了具有慢性感染阶段的丙肝模型,研究表明为了控制疾病应降低与慢性感染者的有效接触率。文[9]中 Stocks 等针对注射吸毒者建立了丙肝传播模型,结果表明针头交换项目和 DAA 治疗可以降低注射吸毒者中丙肝的流行率。文[10]中崔景安等建立了一类具有急慢性感染的丙肝模型并分析了全局稳定性。

但在现实生活中,很多问题具有不确定性,会受到各类随机因素的干扰,确定性传染病模型很难对实际问题进行准确的描述,许多学者研究了具有白噪声干扰的传染病模型^[11-14]。文[15]中研究了一类受随机扰动影响的肺结核模型,发现随机扰动有利于疾病的灭绝。文[16]中建立了具有不同总人口规模的随机 SIQS 传染病模型,利用构造 Lyapunov 函数、伊藤公式得到了疾病的传播阈值,数值模拟的结果表明对接触率的随机扰动有助于减少疾病的传播。

为了更好地描述丙肝的动力学行为,本文在文[10]的基础上稍作改动并加入随机扰动,提出具有随机扰动的 SICR 丙肝传播模型。

1 模型的建立

文[10]中崔景安等建立了如下具有急慢性感染的丙肝模型。

$$\left. \begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \beta SC - aS \\ \frac{dI}{dt} &= \beta SI + \beta SC - (a+r)I \\ \frac{dC}{dt} &= prI - (a+\mu)C \\ \frac{dR}{dt} &= (1-p)rI - aR \end{aligned} \right\} \quad (1)$$

其中: $S(t)$ 、 $I(t)$ 、 $C(t)$ 、 $R(t)$ 分别表示 t 时刻易感者的数量、急性感染者的数量、慢性感染者的数量和恢复者的数量; Λ 表示出生率; β 表示传染率; a 表示自然死亡率; r 表示急性感染者的移出率; p 表示急性感染者转化为慢性感染者的比率; μ 表示慢性丙肝引起的死亡率。

由文[10]知,系统(1)的基本再生数为 $R_0^* = \frac{\beta\Lambda}{a(a+r)} + \frac{\beta\Lambda pr}{a(a+r)(a+\mu)}$, 系统(1)存在唯一的无病平衡点 $E_0^* = \left(\frac{\Lambda}{a}, 0, 0, 0\right)$ 和地方病平衡点 $E_1^* = (S_1^*, I_1^*, C_1^*, R_1^*)$ 。当 $R_0^* < 1$ 时,无病平衡点 E_0^* 存在且全局渐近稳定;当 $R_0^* > 1$ 时,地方病平衡点 E_1^* 存在且全局渐近稳定。其中

$$S_1^* = \frac{\Lambda}{aR_0^*}, I_1^* = \frac{a(a+\mu)}{\beta(a+\mu+pr)}(R_0^* - 1)$$

$$C_1^* = \frac{apr}{\beta(a+\mu+pr)}(R_0^* - 1)$$

$$R_1^* = \frac{(1-p)(a+\mu)r}{\beta(a+\mu+pr)}(R_0^* - 1)$$

考虑到急性感染者与慢性感染者传染率的不同,本文所引用的确定模型在文[10]的基础上稍作改动,用 β_1 表示急性感染者的传染率、用 β_2 表示慢性感染者的传染率,此时有 $R_0 = \frac{\beta_1\Lambda}{a(a+r)} + \frac{\beta_2\Lambda pr}{a(a+r)(a+\mu)}$, 存在唯一的无病平衡点 $E_0 = \left(\frac{\Lambda}{a}, 0, 0, 0\right)$ 和地方病平衡点 $E_1 = (S_1, I_1, C_1, R_1)$ 。其中

$$S_1 = \frac{\Lambda}{aR_0}, I_1 = \frac{a(a+\mu)}{\beta_1(a+\mu) + \beta_2 pr}(R_0 - 1)$$

$$C_1 = \frac{apr}{\beta_1(a+\mu) + \beta_2 pr}(R_0 - 1)$$

$$R_1 = \frac{(1-p)(a+\mu)r}{\beta_1(a+\mu) + \beta_2 pr}(R_0 - 1)$$

类似文[10]的证明,有结论:当 $R_0 < 1$ 时,无病平衡点 E_0 存在且全局渐近稳定;当 $R_0 > 1$ 时,地方病平衡点 E_1 存在且全局渐近稳定。

本文在上述确定模型的基础上,考虑加入噪声的影响,建立如下随机模型

$$\left. \begin{aligned} dS &= [\Lambda - \beta_1 SI - \beta_2 SC - aS]dt + \sigma_1 S dB_1(t) \\ dI &= [\beta_1 SI + \beta_2 SC - (a+r)I]dt + \sigma_2 I dB_2(t) \\ dC &= [prI - (a+\mu)C]dt + \sigma_3 C dB_3(t) \\ dR &= [(1-p)rI - aR]dt + \sigma_4 R dB_4(t) \end{aligned} \right\} \quad (2)$$

式中: $B_1(t)$ 、 $B_2(t)$ 、 $B_3(t)$ 、 $B_4(t)$ 为独立的布朗运动; $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ 为其强度系数。

2 全局正解的存在唯一性

定理 1 对任意给定的初值 $(S(0), I(0), C(0), R(0)) \in R_+^4$, 随机模型存在唯一解 $(S(t), I(t), C(t), R(t)) (t \geq 0)$, 且该解以概率 1 位于 R_+^4 中。

证明: 因为随机模型 (2) 的解满足局部 Lipschitz 连续性条件, 故对任给初值 $(S(t), I(t), C(0), R(0)) \in R_+^4$, 随机模型存在唯一的局部解 $(S(t), I(t), C(t), R(t)), t \in [0, \tau_e), \tau_e$ 是爆破时间^[17]。为证随机模型解的全局存在性, 即只需要证明 $\tau_e = \infty$ 几乎处处成立。取充分大的正整数 n_0 , 使得 $(S(0), I(0), C(0), R(0))$ 均不大于 n_0 , 对于任意正整数 $n \geq n_0$, 定义停时

$$\tau_n = \inf\{t \in [0, \tau_e) : S(t) \geq n; I(t) \geq n; C(t) \geq n; R(t) \geq n\}$$

规定 $\inf \varphi = \infty$ 。由停时的定义知, τ_n 单调递增。不妨设 $\tau_\infty = \lim_{n \rightarrow \infty} \tau_n$, 因此证明解的全局存在性, 只需证明 $\tau_\infty = \infty$ 。

定义函数 $V_1(S, I, C, R) = S + I + C + R$, 由伊藤公式^[18]得

$$\begin{aligned} dV_1 &= LV_1 dt + \sigma_1 S dB_1(t) + \sigma_2 I dB_2(t) + \\ &\sigma_3 C dB_3(t) + \sigma_4 R dB_4(t) \leq \Lambda dt + \sigma_1 S dB_1(t) + \\ &\sigma_2 I dB_2(t) + \sigma_3 C dB_3(t) + \sigma_4 R dB_4(t) \end{aligned} \quad (3)$$

其中: $LV_1 = \Lambda - a(S + I + C + R) - \mu C$ 。

对于任意的 $n \geq n_0$, 存在常数 $T \geq 0$ 使得 $\tau_n \in (0, \tau_n \wedge T)$ 。现对式 (3) 两端从 0 到 $\tau_n \wedge T$ 积分并求期望可得

$$EV_1(S(\tau_n \wedge T), I(\tau_n \wedge T), C(\tau_n \wedge T), R(\tau_n \wedge T)) \leq V(S(0), I(0), C(0), R(0)) + \Lambda T$$

令 $n \rightarrow \infty, t \rightarrow \infty$ 可得到 $\lim_{n \rightarrow \infty} P(\tau_n \leq T) = 0$, 因此 $P(\tau_\infty \leq T) = 0$ 。由于 $T > 0$ 是任意的, 可以得到 $P(\tau_\infty \leq \infty) = 0$ 和 $P(\tau_\infty = \infty) = 1$, 即证 $\tau_\infty = \infty$ 。

这就说明在 $[0, \infty)$ 内 $(S(t), I(t), C(t), R(t))$ 以概率 1 在有限时间内不会产生爆破。

下面两小节将考虑随机模型的解在确定模型的无病平衡点和地方病平衡点附近的渐近行为。

3 当 $R_0 < 1$ 时, 随机模型解的渐近行为

定理 2 若 $R_0 < 1$, 且满足 $\sigma_2^2 < 2a + r - \frac{\beta_1 \Lambda}{a}$,

$\sigma_3^2 < 2(a + \mu) - pr - \frac{\beta_2 \Lambda}{a}, \sigma_4^2 < 2a - (1 - p)r$, 那么对于任意的初值 $(S(0), I(0), C(0), R(0)) \in R_+^4$, 模型 (2) 的解满足

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[\left(S(\xi) - \frac{\Lambda}{a} \right)^2 + I^2(\xi) + \right.$$

$$C^2(\xi) + R^2(\xi) \big] d\xi \leq \frac{\eta}{m}$$

其中

$$\eta = \frac{(\beta_1 + \beta_2) \Lambda^3 + 2a\sigma_1^2 \Lambda^2 + 2a^2 \Lambda^2 + 2a(a + r) \Lambda^2}{2a^3},$$

$$m = \min \left\{ 2a, a + \frac{r}{2} - \frac{\beta_1 \Lambda + a\sigma_2^2}{2a}, \right.$$

$$\left. a + \mu - \frac{\beta_2 \Lambda + apr + a\sigma_3^2}{2a}, a - \frac{(1 - p)r + \sigma_4^2}{2} \right\}.$$

证明: 定义 Lyapunov 函数

$$V_2 = V_{21} + V_{22} + V_{23} + V_{24}$$

其中

$$V_{21} = \frac{1}{2} \left(S - \frac{\Lambda}{a} \right)^2, V_{22} = \frac{1}{2} C^2,$$

$$V_{23} = \frac{1}{2} R^2, V_{24} = \frac{1}{2} \left(S - \frac{\Lambda}{a} + I \right)^2.$$

由伊藤公式得

$$dV_{21} = \left[\left(S - \frac{\Lambda}{a} \right) (\Lambda - \beta_1 SI - \beta_2 SC - aS) + \frac{1}{2} \sigma_1^2 S^2 \right] dt +$$

$$\left(S - \frac{\Lambda}{a} \right) \sigma_1 S dB_1(t) \leq$$

$$\left[-a \left(S - \frac{\Lambda}{a} \right)^2 + \frac{\beta_1 \Lambda}{a} SI + \frac{\beta_2 \Lambda}{a} SC + \frac{1}{2} \sigma_1^2 S^2 \right] dt +$$

$$\left(S - \frac{\Lambda}{a} \right) \sigma_1 S dB_1(t) \leq$$

$$\left[-a \left(S - \frac{\Lambda}{a} \right)^2 + \frac{(\beta_1 + \beta_2) \Lambda}{2a} S^2 + \frac{\beta_1 \Lambda}{2a} I^2 + \frac{\beta_2 \Lambda}{2a} C^2 + \right.$$

$$\left. \frac{1}{2} \sigma_1^2 S^2 \right] dt + \left(S - \frac{\Lambda}{a} \right) \sigma_1 S dB_1(t);$$

$$dV_{22} = \left\{ C[prI - (a + \mu)C] + \frac{1}{2} \sigma_3^2 C^2 \right\} dt +$$

$$\sigma_3 C^2 dB_3(t) \leq$$

$$\left[\frac{pr}{2} I^2 - \left(a + \mu - \frac{pr}{2} - \frac{\sigma_3^2}{2} \right) C^2 \right] dt + \sigma_3 C^2 dB_3(t);$$

$$dV_{23} = \left\{ R[(1 - p)rI - aR] + \frac{1}{2} \sigma_4^2 R^2 \right\} dt +$$

$$\sigma_4 R^2 dB_4(t) \leq$$

$$\left[\frac{(1 - p)r}{2} I^2 + \left(\frac{(1 - p)r}{2} - a + \frac{\sigma_4^2}{2} \right) R^2 \right] dt$$

$$\begin{aligned}
& + \sigma_4 R^2 dB_4(t); \\
dV_{24} = & \left\{ \left(S - \frac{\Lambda}{a} + I \right) \left[\Lambda - aS - (a+r)I \right] + \right. \\
& \left. \frac{1}{2} (\sigma_1^2 S^2 + \sigma_2^2 I^2) \right\} dt + \\
& \left(S - \frac{\Lambda}{a} + I \right) (\sigma_1 S dB_1(t) + \sigma_2 I dB_2(t)) \leq \\
& \left[-a \left(S - \frac{\Lambda}{a} \right)^2 + \frac{\sigma_1^2}{2} S^2 - \left(a+r - \frac{\sigma_2^2}{2} \right) I^2 + \right. \\
& \left. \frac{\Lambda^2}{a} + \frac{\Lambda^2(a+r)}{a^2} \right] dt + \\
& \left(S - \frac{\Lambda}{a} + I \right) (\sigma_1 S dB_1(t) + \sigma_2 I dB_2(t)). \\
\text{则} \\
dV_2 = & dV_{21} + dV_{22} + dV_{23} + dV_{24} \leq \\
& \left[-2a \left(S - \frac{\Lambda}{a} \right)^2 - \left(a + \frac{r}{2} - \frac{\beta_1 \Lambda + a\sigma_2^2}{2a} \right) I^2 - \right. \\
& \left(a + \mu - \frac{\beta_2 \Lambda + apr + a\sigma_3^2}{2a} \right) C^2 - \\
& \left(a - \frac{(1-p)r + \sigma_4^2}{2} \right) R^2 + \\
& \left. \left(\frac{(\beta_1 + \beta_2)\Lambda}{2a} + \sigma_1^2 \right) S^2 + \frac{\Lambda^2}{a} + \frac{\Lambda^2}{a^2} (a+r) \right] dt + \\
& \left(2S + I - \frac{2\Lambda}{a} \right) \sigma_1 S dB_1(t) + \\
& \left(S + I - \frac{\Lambda}{a} \right) \sigma_2 I dB_2(t) + \\
& \sigma_3 C^2 dB_3(t) + \sigma_4 R^2 dB_4(t) \leq \\
& \left[-2a \left(S - \frac{\Lambda}{a} \right)^2 - \left(a + \frac{r}{2} - \frac{\beta_1 \Lambda + a\sigma_2^2}{2a} \right) I^2 - \right. \\
& \left(a + \mu - \frac{\beta_2 \Lambda + apr + a\sigma_3^2}{2a} \right) C^2 - \\
& \left. \left(a - \frac{(1-p)r + \sigma_4^2}{2} \right) R^2 + \eta \right] dt + \\
& \frac{\sigma_1 \Lambda^2}{a^2} dB_1(t) + \frac{\sigma_2 \Lambda^2}{a^2} dB_2(t) + \\
& \frac{\sigma_3 \Lambda^2}{a^2} dB_3(t) + \frac{\sigma_4 \Lambda^2}{a^2} dB_4(t) \quad (4)
\end{aligned}$$

对式(4) 两端分别从0到 t 积分,并取期望得

$$\begin{aligned}
EV_2(t) - EV_2(0) & \leq \\
E \int_0^t & \left\{ -2a \left(S(\xi) - \frac{\Lambda}{a} \right)^2 - \left(a + \frac{r}{2} - \frac{\beta_1 \Lambda + a\sigma_2^2}{2a} \right) I^2(\xi) - \right. \\
& \left. \left(a + \mu - \frac{\beta_2 \Lambda + apr + a\sigma_3^2}{2a} \right) C^2(\xi) - \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \left[a - \frac{(1-p)r + \sigma_4^2}{2} \right] R^2(\xi) \right\} d\xi + E(\eta t) + \\
& \frac{\sigma_1 \Lambda^2}{a^2} E \int_0^t dB_1(t) + \frac{\sigma_2 \Lambda^2}{a^2} E \int_0^t dB_2(t) + \\
& \frac{\sigma_3 \Lambda^2}{a^2} E \int_0^t dB_3(t) + \frac{\sigma_4 \Lambda^2}{a^2} E \int_0^t dB_4(t).
\end{aligned}$$

由于 $\limsup_{t \rightarrow \infty} \frac{\langle \int_0^t dB_i(t), \int_0^t dB_i(t) \rangle}{t} \leq 1 < \infty$, 因此

$$\lim_{t \rightarrow \infty} \frac{\int_0^t dB_i(t)}{t} = 0, i = 1, 2, 3, 4.$$

即证

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[\left(S(\xi) - \frac{\Lambda}{a} \right)^2 + I^2(\xi) + C^2(\xi) + R^2(\xi) \right] d\xi \leq \frac{\eta}{m}$$

注1:根据定理2,在满足上述条件的情况下,随机模型将在确定模型的无病平衡点附近振荡,且波动大小与 $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ 成正相关。

4 当 $R_0 > 1$ 时,随机模型解的渐近行为

定理3 设任意 $(S(t), I(t), C(t), R(t)) \in R_+^4$ 是随机模型满足任意初值的正解,当 $R_0 > 1$ 时,有

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[\frac{a}{2} e_4 (S(\xi) - S_1)^2 + \right. \\
& \left. \frac{a+r}{4} e_5 (I(\xi) - I_1)^2 + \frac{a+\mu}{2} e_6 (C(\xi) - C_1)^2 + \right. \\
& \left. \frac{a}{2} e_7 (R(\xi) - R_1)^2 \right] d\xi \leq \delta
\end{aligned}$$

其中

$$\begin{aligned}
\delta = & \sigma_1^2 \left[\frac{e_1}{2} S_1 + \frac{(e_4 + e_5) \Lambda^2}{2a^2} \right] + \sigma_2^2 \left(\frac{e_2}{2} I_1 + \right. \\
& \left. \frac{e_5 \Lambda^2}{2a^2} \right) + \sigma_3^2 \left(\frac{e_3}{2prI_1} C_1 + \frac{e_6 \Lambda^2}{2a^2} \right) + \sigma_4^2 \frac{e_7 \Lambda^2}{2a^2}, \\
e_1 = & \frac{\beta_2 C_1}{\beta_1 I_1 + \beta_2 C_1 + a}, e_2 = 1, e_3 = \beta_2 S_1 C_1, \\
e_4 = & \frac{\beta_1 I_1 + a}{\Lambda}, e_5 = \frac{a(a+r)}{a^2 + (a+r)^2} e_4,
\end{aligned}$$

$$e_6 = \frac{(a+r)(a+\mu)}{4p^2 r^2} e_5, e_7 = \frac{a(a+r)}{4(1-p)^2 r^2} e_5.$$

证明:定义函数 $V_{31}, V_{32}, V_{33}, V_{34}, V_{35}, V_{36}, V_{37}$ 分别为

$$V_{31} = S - S_1 - S_1 \ln \frac{S}{S_1}, V_{32} = I - I_1 - I_1 \ln \frac{I}{I_1},$$

$$V_{33} = \frac{1}{prI_1} \left(C - C_1 - C_1 \ln \frac{C}{C_1} \right),$$

$$V_{34} = \frac{1}{2} (S - S_1)^2, V_{35} = \frac{1}{2} (S - S_1 + I - I_1)^2,$$

$$V_{36} = \frac{1}{2} (C - C_1)^2, V_{37} = \frac{1}{2} (R - R_1)^2.$$

由伊藤公式得

$$dV_{31} = \left[\left(1 - \frac{S_1}{S} \right) (\Lambda - \beta_1 SI - \beta_2 SC - aS) + \frac{1}{2} \sigma_1^2 S_1 \right] dt + \sigma_1 (S - S_1) dB_1 =$$

$$\left[\left(1 - \frac{S_1}{S} \right) (\beta_1 S_1 I_1 + \beta_2 S_1 C_1 + aS_1) - \beta_1 SI - \beta_2 SC - aS \right] dt + \frac{1}{2} \sigma_1^2 S_1 dt +$$

$$\sigma_1 (S - S_1) dB_1 =$$

$$\left[(\beta_1 S_1 I_1 + \beta_2 S_1 C_1 + aS_1) \left(2 - \frac{S_1}{S} - \frac{S}{S_1} \right) + \beta_1 S_1 I_1 \left(-\frac{SI}{S_1 I_1} + \frac{I}{I_1} + \frac{S}{S_1} - 1 \right) + \beta_2 S_1 C_1 \left(-\frac{SC}{S_1 C_1} + \frac{C}{C_1} + \frac{S}{S_1} - 1 \right) + \frac{1}{2} \sigma_1^2 S_1 \right] dt + \sigma_1 (S - S_1) dB_1 =$$

$$\left[-(\beta_1 I_1 + \beta_2 C_1 + a) \frac{(S - S_1)^2}{S} - \beta_1 (S - S_1) (I - I_1) - \beta_2 (S - S_1) (C - C_1) + \frac{1}{2} \sigma_1^2 S_1 \right] dt + \sigma_1 (S - S_1) dB_1;$$

$$dV_{32} = \left\{ \left(1 - \frac{I_1}{I} \right) [\beta_1 SI + \beta_2 SC - (a + r)I] + \frac{1}{2} \sigma_2^2 I_1 \right\} dt + \sigma_2 (I - I_1) dB_2 =$$

$$\left[\left(1 - \frac{I_1}{I} \right) (\beta_1 SI + \beta_2 SC - \beta_1 S_1 I - \frac{\beta_2 S_1 C_1 I}{I_1}) + \frac{1}{2} \sigma_2^2 I_1 \right] dt + \sigma_2 (I - I_1) dB_2 =$$

$$\left[\beta_1 SI - \beta_1 S_1 I - \beta_1 S_1 I + \beta_1 S_1 I_1 \right] + \left(\beta_2 SC - \frac{\beta_2 S_1 C_1 I}{I_1} - \frac{\beta_2 S C I_1}{I} + \beta_2 S_1 C_1 + \beta_2 S_1 C - \beta_2 S_1 C + \beta_2 S C_1 - \beta_2 S C_1 \right) + \frac{1}{2} \sigma_2^2 I_1 \right] dt + \sigma_2 (I - I_1) dB_2 \leq$$

$$\left[\beta_1 (S - S_1) (I - I_1) + \beta_2 (S - S_1) (C - C_1) + \right.$$

$$\beta_2 S_1 C_1 \left(\frac{S}{S_1} - 1 - \ln \frac{S}{S_1} - \ln \frac{C}{C_1} + \ln \frac{I}{I_1} - \frac{I}{I_1} + \frac{C}{C_1} \right) + \frac{1}{2} \sigma_2^2 I_1 \Big] dt + \sigma_2 (I - I_1) dB_2 \leq$$

$$\left[\beta_1 (S - S_1) (I - I_1) + \beta_2 (S - S_1) (C - C_1) + \beta_2 C_1 \frac{(S - S_1)^2}{S} + \beta_2 S_1 C_1 \left(\ln \frac{I}{I_1} - \frac{I}{I_1} - \ln \frac{C}{C_1} + \frac{C}{C_1} \right) + \frac{1}{2} \sigma_2^2 I_1 \right] dt + \sigma_2 (I - I_1) dB_2;$$

$$dV_{33} = \left\{ \frac{1}{prI_1} \left(1 - \frac{C_1}{C} \right) [prI - (a + \mu)C] + \frac{C_1}{2prI_1} \sigma_3^2 \right\} dt + \frac{\sigma_3}{prI_1} (C - C_1) dB_3 \leq$$

$$\left(\frac{I}{I_1} - \frac{C}{C_1} - \ln \frac{I}{I_1} + \ln \frac{C}{C_1} + \frac{C_1}{2prI_1} \sigma_3^2 \right) dt + \frac{\sigma_3}{prI_1} (C - C_1) dB_3;$$

$$dV_{34} = \left[(S - S_1) (\Lambda - \beta_1 SI - \beta_2 SC - aS) + \frac{1}{2} \sigma_1^2 S^2 \right] dt + (S - S_1) \sigma_1 S dB_1 \leq$$

$$\left[-a (S - S_1)^2 - \beta_1 S_1 (S - S_1) (I - I_1) - \beta_2 S_1 (S - S_1) (C - C_1) + \frac{1}{2} \sigma_1^2 S^2 \right] dt + (S - S_1) \sigma_1 S dB_1;$$

$$dV_{35} = \left\{ (S - S_1 + I - I_1) [\Lambda - aS - (a + r)I] + \frac{1}{2} \sigma_1^2 S^2 + \frac{1}{2} \sigma_2^2 I^2 \right\} dt + (S - S_1 + I - I_1) \sigma_1 S dB_1 + (S - S_1 + I - I_1) \sigma_2 I dB_2 \leq$$

$$\left[-\left(a - \frac{(2a + r)^2}{2(a + r)} \right) (S - S_1)^2 - \frac{a + r}{2} (I - I_1)^2 + \frac{1}{2} \sigma_1^2 S^2 + \frac{1}{2} \sigma_2^2 I^2 \right] dt + (S - S_1 + I - I_1) \sigma_1 S dB_1 + (S - S_1 + I - I_1) \sigma_2 I dB_2;$$

$$dV_{36} = \left\{ (C - C_1) [prI - (a + \mu)C] + \frac{1}{2} \sigma_3^2 C^2 \right\} dt + (C - C_1) \sigma_3 C dB_3 \leq$$

$$\left[\frac{p^2 r^2}{2(a + \mu)} (I - I_1)^2 - \frac{a + \mu}{2} (C - C_1)^2 + \frac{1}{2} \sigma_3^2 C^2 \right] dt + (C - C_1) \sigma_3 C dB_3;$$

$$dV_{37} = \left\{ (R - R_1) [(1-p)rI - aR] + \frac{1}{2}\sigma_4^2 R^2 \right\} dt + (R - R_1)\sigma_4 R dB_4 \leq \left[\frac{(1-p)^2 r^2}{2a} (I - I_1)^2 - \frac{a}{2} (R - R_1)^2 + \frac{1}{2}\sigma_4^2 R^2 \right] dt + (R - R_1)\sigma_4 R dB_4.$$

因此考虑函数

$$V_3 = e_1 V_{31} + e_2 V_{32} + e_3 V_{33} + e_4 V_{34} + e_5 V_{35} + e_6 V_{36} + e_7 V_{37}$$

则得到

$$dV_3 = \left\{ -\frac{a}{2}e_4 (S - S_1)^2 - \frac{a+r}{4}e_5 (I - I_1)^2 - \frac{a+\mu}{2}e_6 (C - C_1)^2 - \frac{a}{2}e_7 (R - R_1)^2 + \sigma_1^2 \left[\frac{e_1}{2}S_1 + \frac{(e_4 + e_5)\Lambda^2}{2a^2} \right] + \sigma_2^2 \left(\frac{e_2}{2}I_1 + \frac{e_5\Lambda^2}{2a^2} \right) + \sigma_3^2 \left(\frac{e_3}{2prI_1}C_1 + \frac{e_6\Lambda^2}{2a^2} \right) + \sigma_4^2 \frac{e_7\Lambda^2}{2a^2} \right\} dt + e_1\sigma_1(S - S_1)dB_1 + e_2\sigma_2(I - I_1)dB_2 + e_3\frac{\sigma_3}{prI_1}(C - C_1)dB_3 + e_4(S - S_1)\sigma_1SdB_1 + e_5(S - S_1 + I - I_1)\sigma_1SdB_1 + e_5(S - S_1 + I - I_1)\sigma_2IdB_2 + e_6(C - C_1)\sigma_3CdB_3 + e_7(R - R_1)\sigma_4RdB_4 \quad (5)$$

对式(5) 两端分别从 0 到 t 积分,并取期望得

$$EV_3(t) - EV_3(0) \leq E \int_0^t \left\{ -\frac{a}{2}e_4 (S(\xi) - S_1)^2 - \frac{a+r}{4}e_5 (I(\xi) - I_1)^2 - \frac{a+\mu}{2}e_6 (C(\xi) - C_1)^2 - \frac{a}{2}e_7 (R(\xi) - R_1)^2 + \delta \right\} d\xi + E(q_1M_1 + q_2M_2 + q_3M_3 + q_4M_4).$$

其中

$$q_1 = \left(e_1 + e_4 \frac{\Lambda}{a} + e_5 \frac{2\Lambda}{a} \right) \sigma_1 \frac{\Lambda}{a} + \left(e_1 + e_4 \frac{\Lambda}{a} + e_5 \frac{\Lambda}{a} \right) \sigma_1 S_1 + e_5 \sigma_1 \frac{\Lambda}{a} I_1, \\ q_2 = \left(e_2 + e_5 \frac{2\Lambda}{a} \right) \sigma_2 \frac{\Lambda}{a} + e_5 \sigma_2 \frac{\Lambda}{a} S_1 + \left(e_2 + e_5 \frac{\Lambda}{a} \right) \sigma_2 I_1, \\ q_3 = \left(\frac{e_3}{prI_1} + e_6 \frac{\Lambda}{a} \right) \sigma_3 \frac{\Lambda}{a} + \left(\frac{e_3}{prI_1} + e_6 \frac{\Lambda}{a} \right) \sigma_3 C_1, \\ q_4 = e_7 \sigma_4 \frac{\Lambda^2}{a^2} + e_7 \sigma_4 \frac{\Lambda}{a} R_1, M_i = \int_0^t dB_i(t).$$

由于 $\limsup_{t \rightarrow \infty} \frac{\langle M_i, M_i \rangle_t}{t} \leq 1 < \infty$, 因此 $\lim_{t \rightarrow \infty} \frac{M_i}{t} = 0, i =$

1, 2, 3, 4。

即证

$$\limsup_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[\frac{a}{2}e_4 (S(\xi) - S_1)^2 + \frac{a+r}{4}e_5 (I(\xi) - I_1)^2 + \frac{a+\mu}{2}e_6 (C(\xi) - C_1)^2 + \frac{a}{2}e_7 (R(\xi) - R_1)^2 \right] d\xi \leq \delta$$

注 2: 根据定理 3, 随机模型将在确定模型的地方病平衡点 E_1 附近振荡, 且波动大小与 $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ 成正相关。

5 疾病的平均持续性

定理 4 若 $R_0 > 1$, 且满足 $\min \left\{ \frac{a}{2}e_4 S_1^2, \frac{a+r}{4}e_5 I_1^2, \frac{a+\mu}{2}e_6 C_1^2, \frac{a}{2}e_7 R_1^2 \right\} \geq \delta$ 时, 随机模型的解是平均持续的, 即随机模型的解满足

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(\xi) d\xi > 0, \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(\xi) d\xi > 0, \\ \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t C(\xi) d\xi > 0, \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(\xi) d\xi > 0.$$

证明: 令 $n = \min \left\{ \frac{a}{2}e_4, \frac{a+r}{4}e_5, \frac{a+\mu}{2}e_6, \frac{a}{2}e_7 \right\}$, 因为 $R_0 > 1$ 时, 由定理 3 得

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S(\xi) - S_1)^2 d\xi \leq \frac{\delta}{n}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (I(\xi) - I_1)^2 d\xi \leq \frac{\delta}{n}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (C(\xi) - C_1)^2 d\xi \leq \frac{\delta}{n}, \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t (R(\xi) - R_1)^2 d\xi \leq \frac{\delta}{n}.$$

注意到 $2S_1^2 - 2SS_1 = 2S_1(S_1 - S) \leq S_1^2 + (S - S_1)^2$,

即 $S \geq \frac{S_1}{2} - \frac{(S - S_1)^2}{2S_1}$, 因此有

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(\xi) d\xi = \frac{S_1}{2} - \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{(S(\xi) - S_1)^2}{2S_1} d\xi \geq \frac{S_1}{2} - \frac{\delta}{2S_1 n} > 0.$$

同理得

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(\xi) d\xi =$$

$$\frac{I_1}{2} - \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{(I(\xi) - I_1)^2}{2I_1} d\xi \geqslant$$

$$\frac{I_1}{2} - \frac{\delta}{2I_1 n} > 0;$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t C(\xi) d\xi =$$

$$\frac{C_1}{2} - \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{(C(\xi) - C_1)^2}{2C_1} d\xi \geqslant$$

$$\frac{C_1}{2} - \frac{\delta}{2C_1 n} > 0;$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t R(\xi) d\xi =$$

$$\frac{R_1}{2} - \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{(R(\xi) - R_1)^2}{2R_1} d\xi \geqslant$$

$$\frac{R_1}{2} - \frac{\delta}{2R_1 n} > 0.$$

所以,随机模型的解是平均持续的。

6 疾病的灭绝性

在传染病模型中,比起疾病的持续性,什么情况下疾病会灭绝是人们更关注的问题,下面将考虑疾病灭绝的条件。

引理 1^[19] 设随机模型的解满足初始条件 $(S(0), I(0), C(0), R(0)) \in R_+^4$, 则有以下结论

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{I(t)}{t} = 0,$$

$$\lim_{t \rightarrow \infty} \frac{C(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{R(t)}{t} = 0.$$

和

$$\lim_{t \rightarrow \infty} \frac{\int_0^t S(s) dB_1(s)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t I(s) dB_2(s)}{t} = 0,$$

$$\lim_{t \rightarrow \infty} \frac{\int_0^t C(s) dB_3(s)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\int_0^t R(s) dB_4(s)}{t} = 0.$$

定义

$$R_0^s = \frac{3\beta\Lambda}{a \left[\left(a + \frac{\sigma_2^2}{2} \right) \wedge \left(a + \mu + \frac{\sigma_3^2}{2} \right) \wedge \left(a + \frac{\sigma_4^2}{2} \right) \right]}.$$

定理 5 当 $R_0^s < 1$ 时,则随机模型的解满足

$$\lim_{t \rightarrow \infty} \langle S \rangle_t = \frac{\Lambda}{a}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln[I(t) + C(t) + R(t)] \leqslant \frac{\beta\Lambda}{a} -$$

$$\frac{1}{3} \left[\left(a + \frac{\sigma_2^2}{2} \right) \wedge \left(a + \mu + \frac{\sigma_3^2}{2} \right) \wedge \left(a + \frac{\sigma_4^2}{2} \right) \right] < 0$$

证明: 令 $W(t) = I(t) + C(t) + R(t)$, $\beta = \max\{\beta_1, \beta_2\}$ 由伊藤公式得

$$d \ln W(t) = LV \ln W(t) + \frac{1}{I + C + R}$$

$$(\sigma_2 I dB_2(t) + \sigma_3 C dB_3(t) + \sigma_4 R dB_4(t))$$

其中

$$LV \ln W(t) = \frac{\beta_1 SI + \beta_2 SC - aI - (a + \mu)C - aR}{I + C + R} -$$

$$\frac{\sigma_2^2 I^2 + \sigma_3^2 C^2 + \sigma_4^2 R^2}{2(I + C + R)^2} \leqslant$$

$$\beta S - \frac{1}{(I + C + R)^2} \left[\left(a + \frac{\sigma_2^2}{2} \right) I^2 + \right.$$

$$\left. \left(a + \mu + \frac{\sigma_3^2}{2} \right) C^2 + \left(a + \frac{\sigma_4^2}{2} \right) R^2 \right] \leqslant$$

$$\beta \frac{\Lambda}{a} - \frac{1}{3} \left[\left(a + \frac{\sigma_2^2}{2} \right) \wedge \left(a + \mu + \frac{\sigma_3^2}{2} \right) \wedge \left(a + \frac{\sigma_4^2}{2} \right) \right]$$

(6)

对式(6) 两端分别从 0 到 t 积分, 结合 $R_0^s < 1$ 得到

$$\limsup_{t \rightarrow \infty} \frac{\ln W(t)}{t} < 0,$$

进一步得到

$$\lim_{t \rightarrow \infty} I(t) = 0, \lim_{t \rightarrow \infty} C(t) = 0, \lim_{t \rightarrow \infty} R(t) = 0.$$

另一方面,由随机模型可以得到

$$d(S + I + C + R) = [\Lambda - a(S + I + C + R) - \mu C] dt + \sigma_1 S dB_1 + \sigma_2 I dB_2 + \sigma_3 C dB_3 + \sigma_4 R dB_4$$

则

$$\frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} +$$

$$\frac{C(t) - C(0)}{t} + \frac{R(t) - R(0)}{t} =$$

$$\Lambda - a \langle S \rangle_t - a \langle I \rangle_t - (a + \mu) \langle C \rangle_t -$$

$$a \langle R \rangle_t + \frac{\sigma_1}{t} \int_0^t S(r) dB_1(r) +$$

$$\frac{\sigma_2}{t} \int_0^t I(r) dB_2(r) + \frac{\sigma_3}{t} \int_0^t C(r) dB_3(r) +$$

$$\frac{\sigma_4}{t} \int_0^t R(r) dB_4(r).$$

相当于

$$\langle S \rangle_t = \frac{\Lambda}{a} - \langle I \rangle_t - \frac{a + \mu}{a} \langle C \rangle_t - \langle R \rangle_t + \varphi(t).$$

其中

$$\varphi(t) = \frac{1}{a} \left[\frac{\sigma_1}{t} \int_0^t S(r) dB_1(r) + \frac{\sigma_2}{t} \int_0^t I(r) dB_2(r) + \right.$$

$$\left. \frac{\sigma_3}{t} \int_0^t C(r) dB_3(r) + \frac{\sigma_4}{t} \int_0^t R(r) dB_4(r) - \frac{S(t) - S(0)}{t} - \right.$$

$$\left[\frac{I(t) - I(0)}{t} - \frac{C(t) - C(0)}{t} - \frac{R(t) - R(0)}{t} \right]_0.$$

由引理 1 知 $\lim_{t \rightarrow \infty} \varphi(t) = 0$, 从而进一步得到 $\lim_{t \rightarrow \infty} \langle S \rangle_t =$

$$\frac{\Lambda}{a}.$$

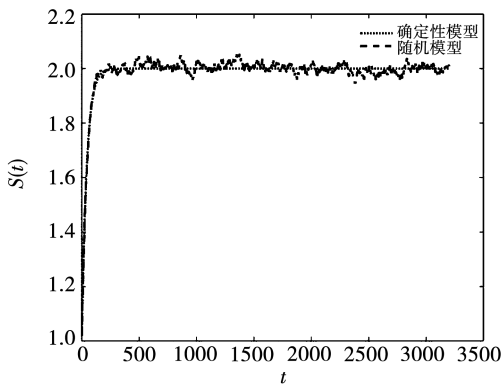
注 3: 确定性系统的基本再生数控制着疾病的存在或灭绝, 从定理 5 的证明得到对于随机系统也存在着阈值 R_0^S , 当 $R_0^S < 1$ 时疾病灭绝。

7 数值模拟

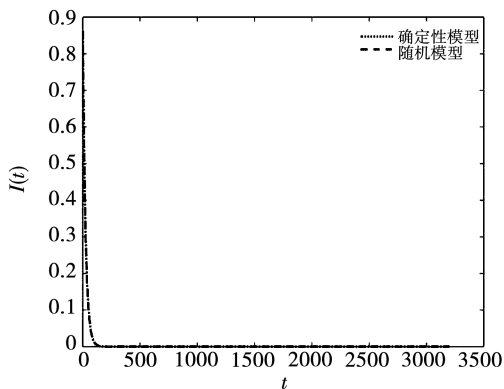
受文[12]和[20]的启发, 借助 Matlab 模拟确定模型和随机模型的曲线, 来说明噪声对丙肝传播的影响。

1) 选取初值 $(S(0), I(0), C(0), R(0)) = (1, 0.9, 0.6, 0.5)$, 取参数 $\Lambda = 1, \beta_1 = 0.0015, \beta_2 = 0.002, a = 0.5, r = 0.4, p = 0.8, \mu = 0.1, \sigma_1 = 0.009, \sigma_2 = 0.001, \sigma_3 = 0.001, \sigma_4 = 0.001$, 计算得到 $R_0 < 1$, 并且满足定理 2 的条件, 则模拟结果如图 1(a) ~ (c) 所示。当取 $\sigma_1 = 0.02, \sigma_2 = 0.02, \sigma_3 = 0.03, \sigma_4 = 0.04$, 其它参数值不变时, 则模拟结果如图 1(d) ~ (f) 所示。确定模型和随机模型的解均收敛于 $E_0 = \left(\frac{\Lambda}{a}, 0, 0, 0\right)$, 并且随机模型的振幅与 $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ 成正相关。

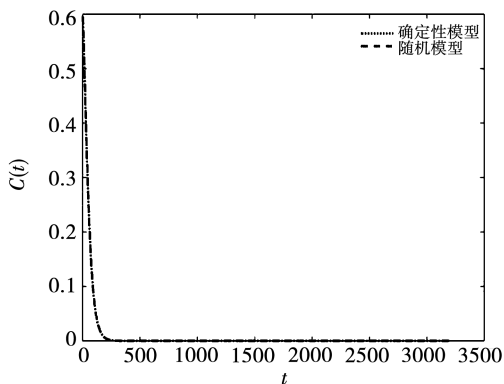
2) 当参数取 $\Lambda = 4, \beta_1 = 0.015, \beta_2 = 0.02, a = 0.1, r = 0.4, p = 0.8, \mu = 0.1, \sigma_1 = 0.009, \sigma_2 = 0.001, \sigma_3 = 0.001, \sigma_4 = 0.001$, 此时 $R_0 > 1$, 得到图 2(a) ~ (c); 当取 $\sigma_1 = 0.09, \sigma_2 = 0.03, \sigma_3 = 0.02, \sigma_4 = 0.02$, 其它参数值不变时, 则模拟结果如图 2(d) ~ (f) 所示。由图 2 可以看出, 此时随机模型和确定模型都收敛于 $E_1 = (S_1, I_1, C_1, R_1)$, 并且随机模型的振幅与 $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ 成正相关。



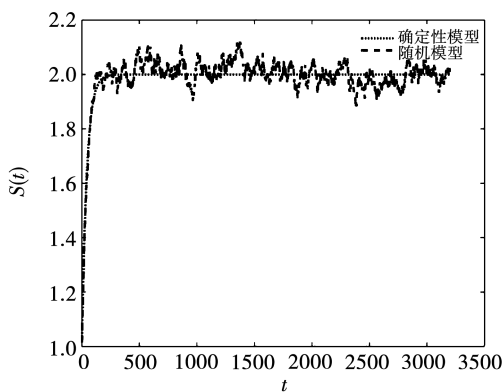
(a) 噪声较小时, 易感者的渐近行为



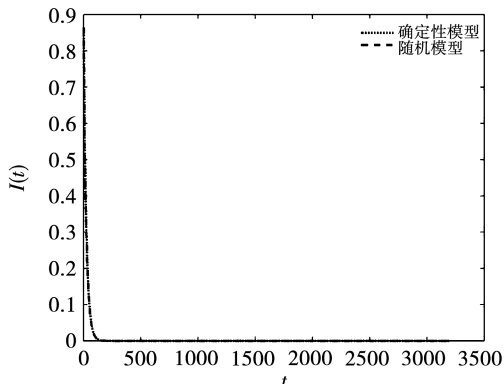
(b) 噪声较小时, 急性感染者的渐近行为



(c) 噪声较小时, 慢性感染者的渐近行为



(d) 噪声较大时, 易感者的渐近行为



(e) 噪声较大时, 急性感染者的渐近行为

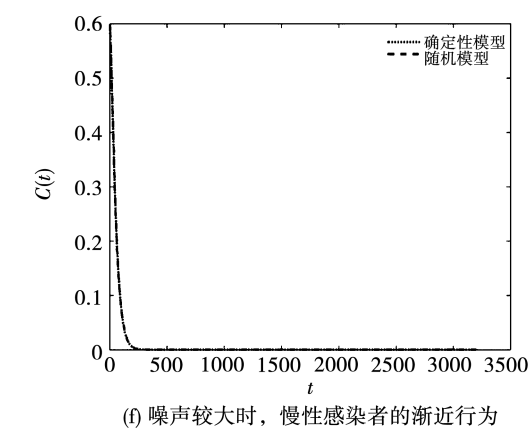


图 1 当 $R_0 < 1$, 随机模型与确定性模型的解的渐近行为

Fig.1 The asymptotic behavior of solutions of stochastic model And deterministic model when $R_0 < 1$

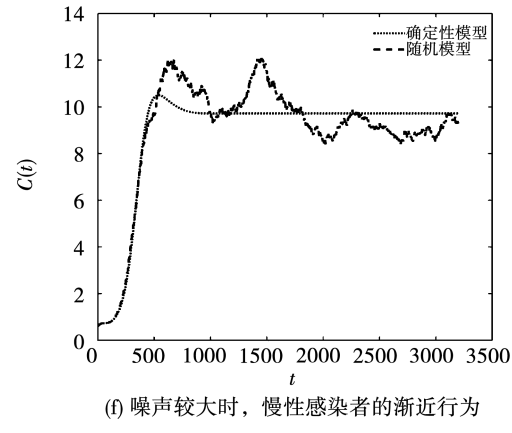
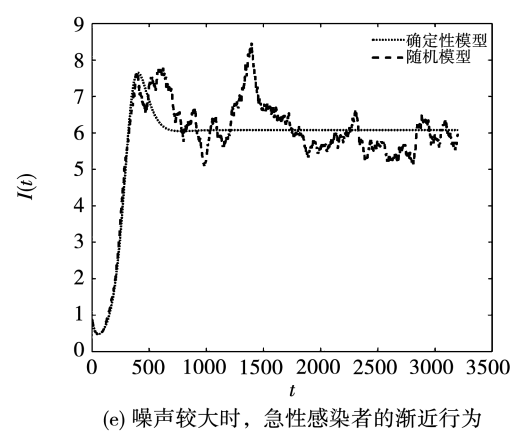
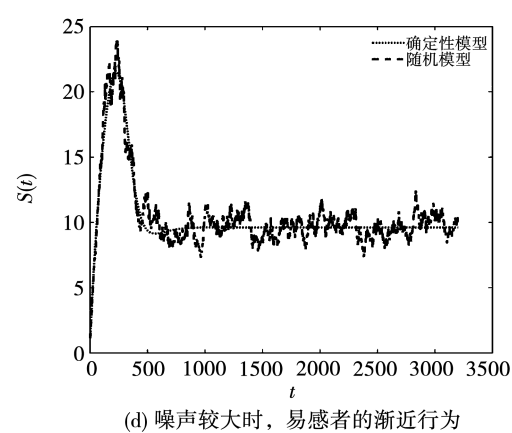
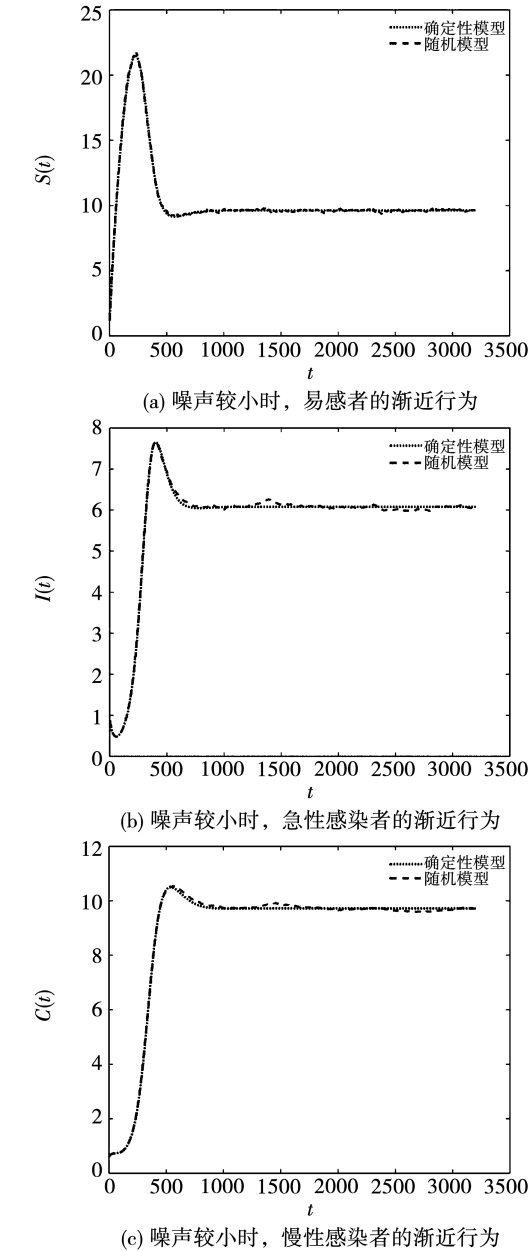


图 2 当 $R_0 > 1$, 随机模型与确定性模型的解的渐近行为

Fig.2 The asymptotic behavior of solutions of stochastic model and deterministic model when $R_0 > 1$

3) 当参数取 $\Lambda = 4, \beta_1 = 0.015, \beta_2 = 0.02, a = 0.1, r = 0.4, p = 0.8, \mu = 0.1, \sigma_1 = 0.9, \sigma_2 = 2.2, \sigma_3 = 2, \sigma_4 = 2.3$, 此时 $R_0 > 1, R_0^s < 1$, 得到图 3。从图 3 可以看出, 虽然确定性模型的解仍在地方病平衡点处稳定, 但随机模型中的丙肝已经灭绝, 由此得到当噪声足够大时, 可导致疾病灭绝。

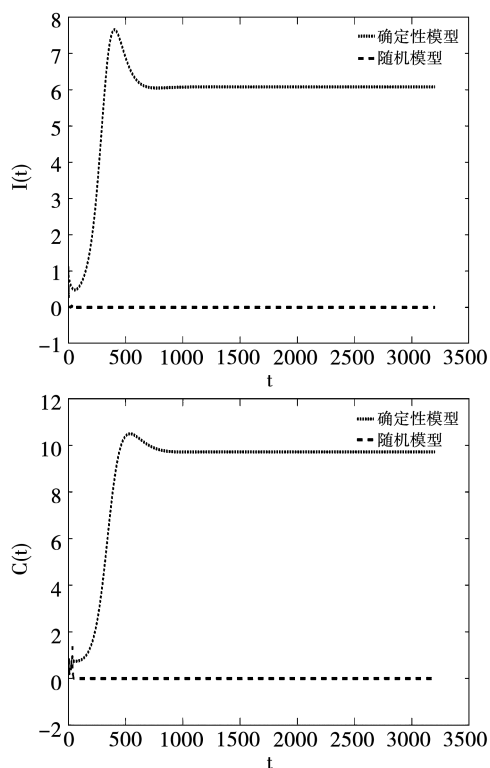


图3 强噪声时,随机模型与确定性模型的解曲线

Fig.3 The solution curves of the stochastic model and deterministic model in strong noise

8 结 论

本文研究了具有急性慢性阶段的随机 SICR 丙肝模型,首先得到了模型全局正解的存在唯一性和有界性,接着证明了当 $R_0 < 1$,随机模型在确定性模型的无病平衡点附近扰动,且扰动强度与 $\sigma_i, i = 1, 2, 3, 4$ 成正相关;证明了当 $R_0 > 1$,随机模型在确定性模型的正平衡点附近扰动,且扰动强度与 $\sigma_i, i = 1, 2, 3, 4$ 成正相关。最后讨论了丙肝持续和灭绝的条件,并通过数值模拟考虑了噪声对随机模型渐近性的影响,进一步证明了上面的理论结果。

但由于丙肝的传播机理的复杂性以及在现实生活中会受到时滞和其他未知因素的影响,下一步的工作可以构建更贴切实际的丙肝模型并在其基础上考虑其他形式的随机丙肝模型,分析其动力学性质,甚至进一步考虑同时具有时滞和随机影响的丙肝模型也是非常必要的,这些因素我们将在未来的工作中考虑。

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